Name:	Date:	Teacher:

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St John's College

Level 3 Mathematics 2014

Apply the Algebra of Complex Numbers in Solving Problems

91577

Trial – Actual is 5 Credits

You should answer ALL the questions in this booklet.

Show ALL working.

If you need more space for any answer, use a the back page and clearly number the question.

Achievement Criteria	For Assessor's use only	
Achievement	Achievement with Merit	Achievement with Excellence
.	.	.
Overall Level of Performance		

Assessor's use only

You are advised to spend 60 minutes answering the questions in this booklet.

QUESTION ONE

(a) Write $\frac{4}{3-\sqrt{12}}$ in the simplest form $a+b\sqrt{c}$ where a, b and c are rational numbers.

(b) Find x and y if $(4 - 2i)(x + iy) = 26 + 2i$

(c) Solve for x, giving exact solutions: $\sqrt{3x+1}+2=x$,.

(d) Describe the geometric locus of points that satisfy the constraint $|z + 1| = |z - i|$ and give the Cartesian equation of the points.

(e) Solve for z: $z^4 + z^2 = -1$

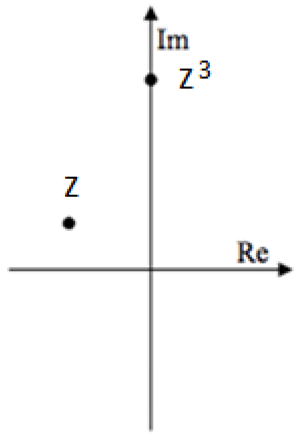
Assessor's use only

Assessor's use only

QUESTION TWO

(a) If $w = 2A \text{ cis } \frac{\pi}{6}$ find w^{10} in exact polar form.

(b) Complex number z has modulus greater than 1.
Both z and z^3 are shown on the Argand diagram opposite, with z^3 on the imaginary axis.



(i) Show where z^2 would be on the diagram.

(ii) Find the argument of z.

(c) z is the complex number $z = 1 + \sqrt{3}\, i$. Find the integer values n for which z^n is purely real.

(d) The number 2 is a real root of $z^3 = 8$. Find the distance $|2 - w|$ if w is a complex root of $z^3 = 8$ leaving your answer in surd form.

Assessor's use only

Assessor's use only

QUESTION THREE

(a) (i) Find the value of k if the polynomial $p(x) = x^3 + 7x^2 + 17x + k$ is divisible by $(x + 3)$.

(ii) Using the value of k obtained in (i) factorise the polynomial $p(x)$ completely.

(b) Find z if $\arg(z - 4) = \frac{\pi}{2}$ and $|z| = 5$

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(c) The complex number $z = -1 + 2i$ is a root of $z^3 - z^2 - z - a = 0$ where a is a real number. Find a and the other roots.

(d) Solve for z: $z^2 + \bar{z} = 0$ giving your answer(s) in the form $a + bi$.

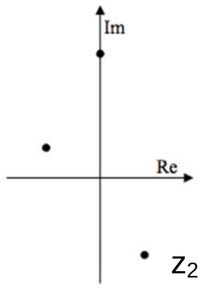
Extra space if required.
Clearly number the question (if required).

Assessor's use only

Question number	

ASSESSMENT SCHEDULE

One	Answer	Achieved (u)	Merit (r)	Excellence (t)
(a)	$\frac{4}{3-\sqrt{12}} \times \frac{3+\sqrt{12}}{3+\sqrt{12}}$ $= \frac{12+4\sqrt{12}}{9-12} \quad = \frac{12+8\sqrt{3}}{-3}$ $= {}^{-}4 + \frac{-8}{3}\sqrt{3}$ or $\frac{-12-8\sqrt{3}}{3}$	Answer given.		
(b)	$4x + 4yi - 2xi - 2yi^2 = 26 + 2i$ $4x + 2y = 26$ real part ${}^{-}2x + 4y = 2$ imaginary part $x = 5, y = 3$	Correct values for x and y		
(c)	$3x + 1 = (x - 2)^2$ $3x + 1 = x^2 - 4x + 4$ $x^2 - 7x + 3.5^2 = {}^{-}3 + 3.5^2$ $(x - 3.5)^2 = \pm\sqrt{9.25}$ $x = 3.5 \pm \sqrt{9.25}$ (0.45861 and 6.5413) Checking $\sqrt{3 \times 0.45 + 1} + 2 \neq 0.45$ $\sqrt{3 \times 6.5413 + 1} + 2 \approx 6.5413$ Answer: $x = 3.5 + \sqrt{9.25}$ only	Gets both correct solutions to quadratic , including in non-exact form	Correct exact solution given after checking Also accept $x = \frac{7+\sqrt{37}}{2}$ and similar variants found via quadratic formula	
(d)	The distance between z and -1 is the same as the distance from z to *i*. <u>or</u> the perpendicular bisector of the line between -1 and *i*. $\lvert (x+1) + yi \rvert = \lvert x + (y-1)i \rvert$ $(x+1)^2 + y^2 = x^2 + (y-1)^2$ $x^2+2x+1 + y^2 = x^2+y^2-2y+1$ $y = {}^{-}x$	Gives either correct equation or geometric interpretation	Gives correct equation and geometric interpretation	
(e)	Let $z = x^2$ so $z^2 + z + 1 = 0$ Then $(z + ½)^2 = {}^{-}¾$ $z = {}^{-}0.5 + \sqrt{0.75}\,i$ $z = 1 \text{ cis } \frac{2\pi}{3}$ and $1 \text{ cis } \frac{4\pi}{3}$ need to take square roots of z to find x, using De Moivre $x = \text{cis}\frac{\pi}{3}, \text{cis}\frac{4\pi}{3}, \text{cis}\frac{2\pi}{3}, \text{cis}\frac{5\pi}{3}$	Find roots to $x^4 + x^2 + 1 = 0$	Recognise need to take roots of z to get x, but only get 2 correct roots	All four roots Alternatively $x = \frac{1 \pm \sqrt{3}\,i}{2}$ and $\frac{-1 \pm \sqrt{3}\,i}{2}$

Two	Answer	Achieved (u)	Merit (r)	Excellence (t)
(a)	$w^{10} = 1024\,A^{10}\ \text{cis}\,\frac{5\pi}{3}$	answer given or $2^{10}A^{10}$ cis $\frac{10\pi}{6}$		
(b)(i)		Point correctly placed in 4th quadrant with modulus between that of z and z^3.		
(b)(ii)	$z^3 = ki$ so $z^3 = k$ cis $(\pi/2)$ $z = \sqrt[3]{k}$ cis $(\pi/6 + n\,2\pi/3)$ In context $\arg(z) = \frac{5\pi}{6}$		Correct argument	
(c)	$z = 1 + \sqrt{3}\,i \quad = 2\ \text{cis}\,\frac{\pi}{3}$ $z^n = 2^n\ \text{cis}\,\frac{n\pi}{3}$ which is real if $\frac{n\pi}{3}$ is a multiple of π so n is any multiple of 3	Any one correct answer	n is a multiple of 6 (i.e. forgot the negative reals)	Answer
(d)	$z^3 = 8 = 8$ cis 0 $z = 2$ cis 0, 2 cis $\frac{2\pi}{3}$, cis $\frac{4\pi}{3}$ $z = 2, {}^-1 \pm \sqrt{3}\,i$ but the first is not complex $\lvert 2 - w \rvert = \lvert 2 - {}^-1 + \sqrt{3}\,i \rvert$ $= \sqrt{3^2 + \sqrt{3}^2} = \sqrt{12} = 2\sqrt{3}$ Or the $\lvert 2 - w \rvert$ length is one side of the equilateral triangle formed by connecting the 3 roots on the Argand diagram.		de Moivre correct for solutions of z,	Answer given

Three	Answer	Achieved	Merit	Excellence
(a)(i)	$p(^{-}3) = 0$ by Remainder Th $(^{-}3)^3 + 7(^{-}3)^2 + 17(^{-}3) + k = 0$ $k = 15$	Answer given.		
(a)(ii)	$x^3 + 7x^2 + 17x + 15 = 0$ which solves on graphics to $x = {^{-}3}, {^{-}2} \pm i$ $(x + 3)(x + 2 + i)(x + 2 - i)$	Correctly factorises (not solves).		
(b)	$\arg(z - 4) = \frac{\pi}{2}$ means the number is on the line $x = 4$ (for $z = x + yi$) and $y > 0$ $\lvert z\rvert = 5$, so $5 = \sqrt{4^2 + y^2}$ $y = 3$	If answer $= \pm 3$	Answer correct	
(c)	$(z + 1 - 2i)(z + 2 + 2i)(z + b)$ $= z^3 - z^2 - z - a$ $(z^2 + 2z + 5)(z + b)$ $= z^3 - z^2 - z - a$ $bz^2 + 2z^2 = {^{-}1}z^2$ (match coeff) $\Rightarrow b = {^{-}3}$ $a = (1 - 2i)(1 + 2i)(^{-}3) = 15$ roots are ${^{-}1} \pm 2i, 3$	Correct value of either a or other roots.	Correct value of a and roots.	
(d)	$z^2 = {^{-}\bar{z}}$ $(x + yi)^2 = {^{-}(x - yi)}$ $x^2 + 2xyi + yi^2 = {^{-}x} + yi$ $x^2 - y^2 = {^{-}x}$ and $2xy = y$ from $2xy = y$, $y = 0$ or $x = ½$ from $x^2 - 0^2 = {^{-}x}$, $x = -1, 0$ or from $0.5^2 - y^2 = {^{-}0.5}$, so $y = \pm\sqrt{0.75}i$ or in polar form $r^2 \text{ cis } 2\theta = {^{-}r} \text{ cis } (^{-}\theta)$ $= r \text{ cis } (^{-}\theta + \pi)$ $\Rightarrow r^2 = r$, so $r = 1$ or 0 and $2\theta = {^{-}\theta} + \pi$ ($3\pi, 5\pi$ etc) $3\theta = \pi, 3\pi, 5\pi$ etc $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ etc, $z = 1 \text{ cis } \frac{\pi}{3}, {^{-}1} (\text{cis } \pi)\ 1 \text{ cis } \frac{5\pi}{3}$,	Any one solution	At least one complex solution	or as fractions ${^{-}1}, 0, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$ 1t if in polar form $0, {^{-}1}, \text{ cis } \frac{\pi}{3}, \text{ cis } \frac{5\pi}{3}$

NØ = No response; no relevant evidence.

N1 = a valid attempt at ONE question.

N2 = ONE question demonstrating limited knowledge of complex number techniques.(1u)

A3 = TWO of u.

A4 = THREE of u.

M5 = ONE of r.

M6 = TWO of r.

E7 = Minor error in 1t

E8 = One of t