90638

ST JOHNS COLLEGE

Level 3 Calculus, 2011

90638 Manipulate real and complex numbers, and solve equations.

Credits: Five

Answer ALL questions in the spaces provided in this booklet.

You should attempt all parts of all questions.

You should show ALL working.

Check that this booklet has pages 2 – 5 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

For Assessor's use only					
Achievement Criteria					
Achievement		**Achievement with Merit**		**Achievement with Excellence**	
Manipulate real and complex numbers, and solve equations.		Solve more complicated equations.		Solve problem(s) involving real or complex numbers.	
Overall level of Performance					

You are advised to spend 60 minutes answering these questions.

QUESTION ONE

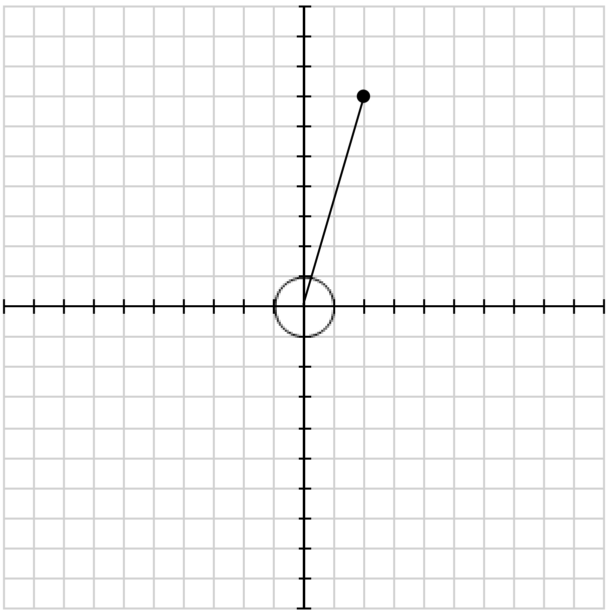
Assessor's use only

(a) Find the values of k such that $2z^2 - 5z + k$ does not give rational roots.

(b) If $u = 2 + 4i$ and $v = 5 - 2i$ find w such that $u\,w = v$

(c) One solution of $z^4 = k$, where k is a complex number, is shown on the Argand diagram below.

Show **all** the other solutions on the diagram.



Assessor's use only

(d) u is a complex number such that $u = 2 + y\,i$ and also $u = k \text{ cis } (1.5)$.

Calculate y and k

(e) $f(x) = 8x^3 + ax^2 + bx + 3$

Calculate a and b if $f(0.5 + 0.5i) = 0$

Assessor's use only

QUESTION TWO

(a) $\frac{7+2\sqrt{3}}{5-\sqrt{3}}$ is written in the form $a + b\sqrt{c}$, where a and b are rational numbers.

Find the values of a and b.

(b) $u = k \text{ cis}\left(\frac{2\pi}{5}\right)$ and $v = 2k \text{ cis}\left(\frac{\pi}{4}\right)$ are two complex numbers.

Find $\frac{u^3}{v}$ in polar form with the argument in terms of π.

(c) If $z^3 = a$ then z is a cube root of a.

Find the cube roots of $1 - i$

(d) Factorise $z^3 - z^2 - z + k = 0$ if $z = 2$ is one root.

Give any complex numbers in the form $z = a + b\sqrt{c}\,i$, where a and b are rational numbers and c is a positive integer.

(e) The locus of a point P in the Argand plane is given by $|z + 2| + |z - 2| = 6$.

Find the points in the Argand plane where P is on the imaginary axis.

Assessor's use only

QUESTION THREE

(a) Solve $3z^2 - 4z + 2 = 0$ giving each solution in the simplest form $z = a + b\sqrt{c}i$, where a and b are rational numbers and c is a positive integer.

(b) Write the equation $3^{2x - k} = 5$ for x in terms of k.

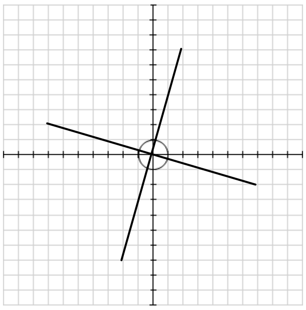
(c) Solve the equation $3\sqrt{x - 10} = x - 20$

Assessor's use only

(d) Solve the equation $3\sqrt{x} + 2 = \sqrt{9x + p}$ for x in terms of p.

(e) z is the complex number $\cos\theta + i\sin\theta$. Prove that $\frac{1+z}{1+\bar{z}} = z$

ASSESSMENT SCHEDULE

No	Evidence	Judgement	Code
1(a)	$az^2 + bz + c$ has complex roots when $b^2 - 4ac < 0$ For $2z^2 - 5z + k$ then $(^-5)^2 - 4 \times 2 \times k < 0$ $k > 3.125$	or $k > \frac{25}{8}$	a
(b)	If u w = v then $w = \frac{v}{u}$ $w = \frac{5-2i}{2+4i} = \frac{5-2i}{2+4i} \cdot \frac{2-4i}{2-4i} = \frac{10-20i-4i+8i^2}{4-8i+8i-16i^2} = \frac{2-24i}{20} = 0.1 - 1.2i$	Or equivalent	a
(c)		no alternatives	m
(d)	$u = 2 + y\,i$ and $u = k$ cis (1.5) k cos (1.5) = 2, so $k = 28.27$ k sin (1.5) = y, so $y = 28.2$	Or equivalent	m
(e)	$f(x) = 8x^3 + ax^2 + bx + 3$ if $f(0.5 + 0.5i) = 0$ $\Rightarrow (x - 0.5 - 0.5i)(x - 0.5 + 0.5i) = x^2 - x + 0.5$ is a factor $(x^2 - x + 0.5)(cx + d) = 8x^3 + ax^2 + bx + 3 \Rightarrow c = 8, d = 6$ $(x^2 - x + 0.5)(8x + 6) = 8x^3 - 2x^2 - 2x + 3 \Rightarrow a = {}^-2, b = {}^-2$ Or $8(0.5 + 0.5i)^3 + a(0.5 + 0.5i)^2 + b(0.5 + 0.5i) + 3 = 0$ $(^-2 + 2i) + 0.5ai + 0.5b + 0.5bi + 3 = 0$ match coefficients $1 + 0.5b = 0$ and $2i + 0.5ai + 0.5bi = 0$ $\Rightarrow b = {}^-2, a = {}^-2$		e

No	Evidence	Judgement	Code
2(a)	$\frac{7+2\sqrt{3}}{5-\sqrt{3}}$ $= \frac{7+2\sqrt{3}}{5-\sqrt{3}} \times \frac{5+\sqrt{3}}{5+\sqrt{3}}$ $= \frac{41+17\sqrt{3}}{22}$ so $a = \frac{41}{22}$ and $b = \frac{17}{22}$		a
(b)	$= \frac{k^3 \text{ cis}\left(\frac{6\pi}{5}\right)}{2k \text{ cis}\left(\frac{\pi}{4}\right)} = \frac{k^3}{2k} \text{cis}\left(\frac{6\pi}{5} - \frac{\pi}{4}\right) = = \frac{k^2}{2} \text{cis}\left(\frac{19\pi}{20}\right)$ or $0.5k^2$ cis $(0.95\,\pi)$ etc		a

(c)	Let z be the complex number $rcis\theta$. Then $z^3 = 1 - i$ means that $r^3 cis3\theta = 1 - i = \sqrt{2}cis\left(\frac{7\pi}{4}\right)$ Equating moduli, $r^3 = \sqrt{2}$ so $r = \sqrt[6]{2}$ Comparing arguments, $3\theta = \frac{7\pi}{4}$, so $\theta = \frac{7\pi}{12}$ then adding $\frac{2\pi}{3} = \frac{8\pi}{12}$ for subsequent roots Hence the cube roots of $1 - i$ are $\sqrt[6]{2}cis\left(\frac{7\pi}{12}\right), \sqrt[6]{2}cis\left(\frac{15\pi}{12}\right), \sqrt[6]{2}cis\left(\frac{23\pi}{12}\right)$	Arguments could be based on $\frac{-\pi}{4}$	m
(d)	$f(2) = 0$ so $2^3 - 2^2 - 2 + k = 0 \Rightarrow k = {}^-2$ $(z - 2)(a z^2 + b z + c) = z^3 - z^2 - z - 2$ Inspection of end terms shows $a = 1$ and $c = 1$, and looking at the z term, ${}^-2b + c = {}^-1$, so $b = 1$ Using quadratic formula to solve $z^2 + z + 1 = 0$ gives $z = \frac{-1\pm\sqrt{1^2-4\times1\times1}}{2\times1} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$ So factorising gives $(z - 2)(z + \frac{1}{2} + \frac{\sqrt{3}}{2} i)(z + \frac{1}{2} - \frac{\sqrt{3}}{2} i)$		m
(e)	On the imaginary axis, the x in $x + yi$ is zero, so $\lvert z + 2\rvert + \lvert z - 2\rvert = \lvert 2 + yi\rvert + \lvert {}^-2 + yi\rvert = 6$ $\sqrt{2^2 + y^2} + \sqrt{(-2)^2 + y^2} = 2\sqrt{4 + y^2} = 6$ $\Rightarrow \sqrt{4 + y^2} = 3$ $\Rightarrow 4 + y^2 = 9$ $y = \pm\sqrt{5}$ and so the points are $\pm\sqrt{5}i$ If you want to do it the long way: Let P be the point $z = x + yi$ $\lvert z + 2\rvert + \lvert z - 2\rvert = 6$ means that $(x + yi) + 2 \rvert + \lvert (x + yi) - 2 \rvert = 6$ $\lvert (x+2) + yi \rvert + \lvert (x-2) + yi) \rvert = 6$ $\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 6$ $\sqrt{(x+2)^2 + y^2} = 6 - \sqrt{(x-2)^2 + y^2}$ $(x+2)^2 + y^2 = 36 - 12\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$ $x^2 + 4x + 4 + y^2 = 36 - 12\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$ $12\sqrt{(x-2)^2 + y^2} = 36 - 8x$ $3\sqrt{(x-2)^2 + y^2} = 9 - 2x$ $9(x^2 - 4x + 4 + y^2) = (9 - 2x)^2 = 81 - 36x + 4x^2$ $5x^2 + 9y^2 = 45$ When $x = 0$, $y^2 = 5$ so $y = \pm\sqrt{5}$ $\therefore$ the points on the imaginary axis are $-\sqrt{5}i$ and $\sqrt{5}i$		e

3(a)	$z = \dfrac{4 \pm \sqrt{16-24}}{6}$ $= \dfrac{4 \pm 2\sqrt{2}i}{6}$ ie $z = \dfrac{2}{3} + \dfrac{\sqrt{2}}{3}i$ or $z = \dfrac{2}{3} - \dfrac{\sqrt{2}}{3}i$	must simplify No alternatives	a
(b)	$3^{2x-k} = 5 \quad \Rightarrow\ 2x - k = \log_3 5 \quad \Rightarrow\ x = \dfrac{\log_3 5 + k}{2}$ or similar alternatives, such as $x = \dfrac{\log 5}{2 \log 3} + \dfrac{k}{2}$		a
(c)	$(3\sqrt{x-10}\,)^2 = (x-20)^2$ $9x - 90 = x^2 - 40x + 400$ $0 = x^2 - 49x + 490$ $x = 14$ or 35 $x = 14$ is invalid as it gives $6 = {}^-6$, so $x = 35$ only solution		m
(d)	$3\sqrt{x} + 2 = \sqrt{9x + p}$ Squaring both sides gives $9x + 12\sqrt{x} + 4 = 9x + p$ ie $12\sqrt{x} = p - 4$ Squaring both sides gives $144x = (p-4)^2$ so $x = \dfrac{(p-4)^2}{144}$	Or equivalent	m
(e)	let $z = a + b\,i$ as $z = 1 \cos\theta$ then $a^2 + b^2 = 1$ $\dfrac{1+z}{1+\bar{z}} = \dfrac{(a+1)+bi}{(a+1)-bi} = \dfrac{((a+1)+bi)}{((a+1)-bi)}\dfrac{((a+1)+bi)}{((a+1)+bi)}$ $= \dfrac{a^2+a+abi+a+1+bi+abi+bi+b^2i^2}{a^2+a+abi+a+1+bi-abi-bi-b^2i^2}$ $= \dfrac{a^2+2a+2abi+1+2bi-b^2}{a+a+1+a^2+b^2} = \dfrac{a^2+2a+1+2abi+2bi-b^2}{2a+2}$ $= \dfrac{(a+1)^2+2bi(a+1)+(a^2-1)}{2(a+1)}$ $\quad a^2 + b^2 = 1$ so ${}^-b^2 = a^2 - 1$ $= \dfrac{(a+1)^2+2bi(a+1)+(a+1)(a-1)}{2(a+1)}$ $= \dfrac{(a+1)+2bi+)+(a-1)}{2} = \dfrac{2a+2bi}{2} = a + bi = z$ The working can be greatly simplified, however, if you rework the original statement so be the equivalent proof that $(z+1) = z\,(\bar{z}+1)$. This leads to far less working.		e